



# Enhanced layerwise model for laminates with imperfect interfaces. Part 1: Equations and theoretical validation

René Alvarez Lima, Alberto Diaz Diaz, Jean-François Caron, Sylvain Chataigner

## ► To cite this version:

René Alvarez Lima, Alberto Diaz Diaz, Jean-François Caron, Sylvain Chataigner. Enhanced layerwise model for laminates with imperfect interfaces. Part 1: Equations and theoretical validation. Composite Structures, 2012, 94 (5), pp. 1694-1702. 10.1016/j.compstruct.2011.12.007 . hal-00851453

**HAL Id: hal-00851453**

**<https://hal.science/hal-00851453>**

Submitted on 14 Aug 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## **Enhanced layerwise model for laminates with imperfect interfaces – Part 1: Equations and theoretical validation**

Alvarez-Lima René<sup>1</sup>, Diaz-Diaz Alberto<sup>1\*</sup>, Caron Jean-François<sup>2</sup>, Chataigner Sylvain<sup>3</sup>

<sup>1</sup>Centro de Investigación en Materiales Avanzados, Miguel de Cervantes 120, C.I.  
Chihuahua, 31109, Chihuahua Mexico.

<sup>2</sup>Université Paris-Est, Institut Navier, LAMI-Ecole des Ponts, 6-8 Av. Blaise Pascal,  
Marne La Vallée, 77455, France.

<sup>3</sup>LUNAM Université. IFSTTAR Centre de Nantes, SOA. Bouguenais, F-443441  
France.

\* Corresponding author. E-mail: [alberto.diaz@cimav.edu.mx](mailto:alberto.diaz@cimav.edu.mx)

Phone: (52-614)439-48-02. Fax: (52-614)439-11-30

### **Abstract**

The aim of this paper is the enhancement and validation of a layerwise model applied to the analysis of laminates with thin layers of an elastic-plastic adhesive. The thin adhesive layers are modeled as imperfect interfaces across which displacement discontinuities exist. In a previous paper, the constitutive equations of the imperfect interfaces were empirically established without following the layerwise logic. The model equations are revisited and a solid theoretical justification of the new enhanced equations is obtained by making use of the Hellinger-Reissner functional. A theoretical validation of the model is performed by comparing its predictions to those of a solid finite element resolution in the case of a T-peel joint. The results of the enhanced version of the model are very accurate whereas those of the previous version are erratic for the considered joint. As compared to the solid finite element method, an important saving in computational cost is achieved.

**Keywords:** Interfaces; Laminates; Stress Analysis; Plasticity; Layerwise Model

## Introduction

Currently, laminated structures are widely applied in several industries. The design of these structures requires the application of operational models for computing stresses, strains and displacements. A 3D finite element calculation of laminated structures may need an excessively high number of elements and the computational cost could become unaffordable, especially when dealing with thin laminates or when strong differences in the thicknesses of the layers exist. This is the case, for example, of thin adhesive layers bonding composite layers in a bonded joint. 2D models are usually a good alternative to the analysis of laminates or joints with a plate or shell topology. In [1], Carrera and Ciuffreda compared several theories of laminated plates and proposed a unified formulation for the development of 2D models. This formulation has been applied to establish the equations of several equivalent single layer or layerwise approaches [2-4]. The governing equations may be obtained by the Reissner's variational method [5,6] or the principle of virtual displacements. As compared to single layer theories, layerwise models represent a more natural method to calculate interfacial stresses and capture specific aspects of interfaces in laminates [7-10] in order to predict delamination or to take into account edge effects.

In a layerwise modelling, stresses or displacements in a layer are approximated by finite series of known  $z$ -functions ( $z$  is the through the thickness coordinate). Carrera classifies these models as axiomatic and does a thorough review of these in [11]. The models differ by the choice of the approximate fields: displacements [12,13] or displacements and stresses [14-16]. Pure stress approaches are less usual. However, the study of stress concentrations or stress controlled phenomena could be more natural, more convenient with a direct description of stress fields. A noteworthy work has been made in this way by Pagano [17] who used Reissner's variational mixed formulation [5] and a stress field approximation to obtain an efficient model. The stress field selection verifies the continuity conditions across the interfaces of the multilayer. The key point which is not often highlighted is that no displacement approximation is made despite the use of a mixed formulation. The Hellinger-Reissner functional and the stress approximation helps to identify 2D generalized displacements, energetically associated to the generalized forces which derived to the stress approximation. No constraint conditions

on the 3D displacement fields are made: “Note that we refrain from assuming the form of the displacement field in accordance with the objectionable features of that approach” ([17], p.389). These considerations lead to a less constraint model than those where both displacement and stress components are approximated. More recent developments of Pagano’s approach can be found in [18-20] where some simplifications are adopted to obtain a more operational layerwise model called M4-5N. In this model, a polynomial approximation of stresses in each layer is proposed and the laminate is modeled by a superposition of Reissner plates [21] coupled with interfacial stresses which ensure the continuity of the stress vector across the interfaces [19]. The M4-5N has already been validated for linear elastic problems [19]. In spite of the wide variety of layerwise models, most of these consider linear elastic materials bonded with perfect interfaces and do not take into account material non-linearities such as plasticity or imperfect interfaces.

In fibre reinforced cross-ply composite laminates, the thin matrix or adhesive layer located at the interface between the plies may exhibit high plastic strains which seem to control delamination onset [22,23]. In a similar manner, in adhesively bonded joints significant plastic strains in the adhesive layers precede by far failure initiation [24]. Polymers such as adhesives or matrices employed in several composite applications may exhibit complex plastic phenomena. Cognard et al. have carried out experiments with adhesive joints which confirm the significant effect of adhesive plasticity on the behavior of the joint [25]. The yield function is sensitive to the hydrostatic pressure; plasticity yielding is better predicted by a Drucker-Prager yield function than a Von-Mises one [26]. Ratcheting may also occur [27] and plasticity may affect the fatigue strength of the adhesive. For all these reasons and since adhesives usually are used to bond the layers in a laminate, it is important to take into account plasticity in the polymer layers of a laminate. When these layers are thin as compared to the structural layers of the laminate, these layers may be modeled as imperfect interfaces in order to reduce the computational cost. Few layerwise models take into account imperfect interfaces and their non-linear constitutive equations.

In [28], Aquino de Los Rios et al proposed an adaptation of the layerwise M4-5N model to analyze laminated structures with elastic-plastic interfaces made up of a thin layer of

an isotropic material. An approximation of the 3D strains and its integration through the thickness of the thin layer provided the displacement discontinuities across the imperfect interface. The equations of the model were solved by means of a Newton-Raphson-like technique and a finite element method in the case of a plane strain state. A first validation of the model and its numerical tool was proposed by comparing the results of the model to those of a 3D finite element model applied to a double lap adhesively bonded joint with an elastoplastic adhesive. The cumulative plastic strains, the interfacial stresses and the displacement discontinuities were accurately predicted by the M4-5N model. In spite of this, Aquino de Los Rios et al remarked the necessity to provide a more rigorous theoretical support of the proposed equations of interlaminar plasticity. This rigorous determination should evoke the Hellinger Reissner functional and the approximation of 3D stresses which is the starting point of the M4-5N model and not the approximation of 3D strains and displacements. Actually, the original elastic model developed in [18,19] does not make any approximation of 3D strains and displacements. Recently, the layerwise model proposed by Aquino de los Rios et al was adopted by Duong et al in [29] to develop a layerwise finite element for laminates with imperfect interfaces in a general 3D case (not only the plane strain state assumed in [28] for the numerical resolution of the equations).

In this paper, a rigorous theoretical support and an enhancement of the equations of interlaminar plasticity obtained by Aquino de los Rios et al [28] are proposed. The enhancement consists on a better description of the stress field in the adhesive and taking into account the out-of-plane Poisson's effect which is neglected in most plate theories. To obtain the equations of the enhanced model of laminates with imperfect interfaces, two steps are proposed (see Figure 1). In the first step, a version of the M4-5N model where the thin adhesive layers are modelled as layers is obtained by applying an asymptotic expansion method. In the second step, the equations in the previous step are arranged to obtain the equations of the model with imperfect interfaces.

In the first part of this article, the equations of the M4-5N model for laminates with thin elastoplastic layers are developed (step 1). Secondly, the previous equations are adapted to obtain an enhancement and a rigorous theoretical basis to the interfacial plastic equations of Aquino de los Rios et al [28]. Finally, a theoretical validation of the

enhanced M4-5N model is performed by comparing its results to those of a solid finite element resolution for the case of a T-peel adhesive joint.

Throughout this work,

- subscripts “,1”, “,2” and “,3” denote the partial derivatives with respect to  $x$ ,  $y$  and  $z$ , respectively
- bold face characters define tensors, matrices and vectors
- subscripts  $o$ ,  $p$ ,  $q$  and  $r$  indicate the components in the  $(x, y, z)$  space; they are assigned the values 1, 2 and 3,
- subscripts  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  indicate the components on the  $(x, y)$  plane and are assigned the values 1 and 2,
- $\mathbf{U}$  and  $\boldsymbol{\sigma}$  denote respectively the 3D displacement field and the 3D stress field,
- each thick layer is orthotropic and one of the orthotropy directions is the  $z$  direction perpendicular to the interfaces between layers
- $\mathbf{S}(x, y, z) = \mathbf{S}(z)$  denotes the 4th-order tensor of compliances; it is constant in each layer. Its components are  $S_{opqr}$ , with  $S_{opqr} = 0$  in the presence of an odd number of 3 ( $z$ -direction) in the set  $opqr$ .

## 1. Modelling a laminate with thick layers bonded by thin layers

In this section, a laminate made up of  $N$  thick layers bonded by  $N-1$  thin layers is considered (see step 1 in Figure 1). The layers are numbered as shown in Figure 1. An odd number corresponds to a thick layer whereas an even number indicates a thin layer. The interfaces between thin and thick layers are perfect. The thick layers are made up of an orthotropic elastic material whereas the thin ones are made up of an elastic-plastic isotropic material.

In this section,

- superscripts  $m$  and  $n, n+1$  indicate layer  $m$  and the interface between layers  $n$  and  $n+1$  ( $1 \leq m \leq 2N-1$ ,  $1 \leq n \leq 2N-2$ ), respectively,

- superscripts  $2k$  and  $2l-1$  indicate a thin layer  $2k$  and a thick layer  $2l-1$  ( $1 \leq k \leq N-1, 1 \leq l \leq N$ )

- the multi-layer in Figure 1 lies within the volume defined by  $\begin{cases} (x, y) \in \Gamma \\ z \in [h_-^1, h_+^{2N-1}] \end{cases}$ ;

- layer  $m$  occupies the geometrical space defined by  $\begin{cases} (x, y) \in \Gamma \\ z \in [h_-^m, h_+^m] \end{cases}$ ; its thickness is

$$e^m = h_+^m - h_-^m,$$

- the thickness of the thin layer  $2k$  is much smaller than the thicknesses of the adjacent thick layers:  $e^{2k} \ll e^{2k-1}$  and  $e^{2k} \ll e^{2k+1}$ ,

- $\Gamma^{n,n+1}$  denotes the interface between layers  $n$  and  $n+1$ ,

- the fourth-order tensors  $\mathbf{S}^m$  and  $\mathbf{S}_Q^m$  represent the in-plane and shearing compliances of layer  $m$ , respectively; they are defined by:

$$S_{\alpha\beta}^m = S_{\alpha\alpha\beta\beta}(z), \quad S_{16}^m = S_{61}^m = 2S_{1112}(z), \quad S_{26}^m = S_{62}^m = 2S_{2212}(z), \quad S_{66}^m = 4S_{1212}(z),$$

$S_{Q\alpha\beta}^m = 4S_{\alpha 3\beta 3}(z)$  for  $z \in [h_-^m, h_+^m]$ . The scalars  $S_3^m$  denote the normal compliance of layer  $m$  and is defined by:  $S_3^m = S_{3333}(z)$  for  $z \in [h_-^m, h_+^m]$

- the compliances related to the coupling between in-plane and normal stresses in layer  $m$  are defined by:  $S_{C\alpha\beta}^m = 2S_{33\alpha\beta}(z)$  for  $z \in [h_-^m, h_+^m]$ .

The steps to follow in the construction of the model are similar to those followed by Pagano [17].

The following basis of third-degree  $z$ -polynomials is defined:

$$\begin{cases} P_0^m(z) = 1 \\ P_1^m(z) = \frac{z - \bar{h}^m}{e^m} \\ P_2^m(z) = -6 \left( \frac{z - \bar{h}^m}{e^m} \right)^2 + \frac{1}{2} \\ P_3^m(z) = -2 \left( \frac{z - \bar{h}^m}{e^m} \right)^3 + \frac{3}{10} \left( \frac{z - \bar{h}^m}{e^m} \right) \end{cases} \quad (1)$$

where  $\bar{h}^m = \frac{h_+^m + h_-^m}{2}$ . In layer  $m$ , the in-plane stress components  $\sigma_{\alpha\beta}$  ( $\alpha, \beta \in \{1, 2\}$ ) are

chosen as a linear combination of  $P_0^m$  and  $P_1^m$  and the 3D equilibrium equations lead

both to shear stresses  $\sigma_{\alpha 3}$  in the form of second-degree polynomials of  $z$  and to the normal stress  $\sigma_{33}$  as a third-degree polynomial. The expressions of these polynomials may be found in [18]. The polynomial coefficients are expressed in terms of the following generalized internal forces [18,19]:

- force, moment and shear resultants of layer  $m$ , respectively :

$$N_{\alpha\beta}^m(x, y) = \int_{h_-^m}^{h_+^m} \sigma_{\alpha\beta}(x, y, z) P_0^m(z) dz, \quad M_{\alpha\beta}^m(x, y) = e^m \int_{h_-^m}^{h_+^m} \sigma_{\alpha\beta}(x, y, z) P_1^m(z) dz \quad (2)$$

$$\text{and } Q_\alpha^m(x, y) = \int_{h_-^m}^{h_+^m} \sigma_{\alpha 3}(x, y, z) P_0^m(z) dz$$

- interfacial shear and peel stresses at interfaces  $\Gamma^{n,n+1}$  :

$$\tau_{\alpha 3}^{n,n+1}(x, y) = \sigma_{\alpha 3}(x, y, h_+^n) = \sigma_{\alpha 3}(x, y, h_-^{n+1}) \quad (3)$$

$$\text{and } \sigma_3^{n,n+1}(x, y) = \sigma_{33}(x, y, h_+^n) = \sigma_{33}(x, y, h_-^{n+1})$$

where  $(x, y) \in \Gamma$ . Let us point out that the generalized interfacial stresses ensure the continuity of the stress vector across the interfaces.

Assuming that volume forces are negligible, the Hellinger-Reissner functional for elastic problems applied to the laminate is:

$$H.R.(\mathbf{U}^*, \boldsymbol{\sigma}^*) = \sum_{m=1}^{2N-1} \int_{\Gamma} \int_{h_-^m}^{h_+^m} \left[ \sigma_{op}^* (\varepsilon_{op}(\mathbf{U}^*) - \varepsilon_{op}^p) - \frac{1}{2} \sigma_{op}^* S_{opqr} \sigma_{qr}^* \right] dz d\omega + F(\mathbf{U}^d, \mathbf{T}^d) \quad (4)$$

where  $\varepsilon_{op}^p$  is the  $op$  component of the plastic strain tensor  $\boldsymbol{\varepsilon}^p$  (it is zero in the thick layers),  $\mathbf{U}^*$  is a piecewise  $C^1$  first order tensor field,  $\boldsymbol{\sigma}^*$  is a piecewise  $C^1$  second order symmetric tensor field,  $F(\mathbf{U}^d, \mathbf{T}^d)$  is a boundary integration term which involves the imposed displacement vector  $\mathbf{U}^d$  and stress vector  $\mathbf{T}^d$  at the boundaries of the structure.



By introducing the stress approximation in the term  $\int_{h_m^-}^{h_m^+} \sigma_{op}^* \varepsilon_{op}(\mathbf{U}^*) dz$  in equation (4) and integrating by parts, one identifies the following  $5N$  generalized displacements for  $(x, y) \in \Gamma$ :

$$\begin{aligned} U_\alpha^m(x, y) &= \int_{h_m^-}^{h_m^+} \frac{P_0^m(z)}{e^m} U_\alpha(x, y, z) dz, \quad \Phi_\alpha^m(x, y) = \int_{h_m^-}^{h_m^+} \frac{12 P_1^m(z)}{e^{m^2}} U_\alpha(x, y, z) dz \\ \text{and } U_3^m(x, y) &= \int_{h_m^-}^{h_m^+} \frac{P_0^m(z)}{e^m} U_3(x, y, z) dz \end{aligned} \quad (5).$$

The term  $\int_{h_m^-}^{h_m^+} \sigma_{op}^* \varepsilon_{op}(\mathbf{U}^*) dz$  then provides the following generalized strains:

$$\begin{aligned} \varepsilon_{\alpha\beta}^m(x, y) &= \frac{1}{2} (U_{\alpha,\beta}^m + U_{\beta,\alpha}^m), \quad \chi_{\alpha\beta}^m(x, y) = \frac{1}{2} (\Phi_{\alpha,\beta}^m + \Phi_{\beta,\alpha}^m), \\ d_{\Phi_\alpha}^m(x, y) &= \Phi_\alpha^m + U_{3,\alpha}^m, \quad D_\alpha^{n,n+1}(x, y) = U_\alpha^{n+1} - U_\alpha^n - \frac{e^n}{2} \Phi_\alpha^n - \frac{e^{n+1}}{2} \Phi_\alpha^{n+1} \\ \text{and } D_3^{n,n+1}(x, y) &= U_3^{n+1} - U_3^n \end{aligned} \quad (6)$$

Let us point out that no approximation of the 3D displacement is proposed in our model. As already mentioned in the introduction, this is an important difference with other layerwise models such as those proposed in [14,15].

The variational property of the  $H.R.$  functional in equation (4) with respect to the generalized displacements of the thick layers yields the generalized equilibrium and boundary conditions established in the original elastic model [19]. These equilibrium equations are:

$$\begin{cases} N_{\alpha\beta,\beta}^{2l-1} + \tau_\alpha^{2l-1,2l} - \tau_\alpha^{2l-2,2l-1} = 0 \\ Q_{\alpha,\alpha}^{2l-1} + \sigma_3^{2l-1,2l} - \sigma_3^{2l-2,2l-1} = 0 \\ M_{\alpha\beta,\beta}^{2l-1} - Q_\alpha^{2l-1} + \frac{e^{2l-1}}{2} (\tau_\alpha^{2l-1,2l} + \tau_\alpha^{2l-2,2l-1}) = 0 \end{cases} \quad (7)$$

For the thin layers, an asymptotic analysis using the small thickness of these layers proves that no boundary conditions are to be considered and the generalized equilibrium equations are:

$$\begin{cases} \tau_1^{2k-1,2k} = \tau_1^{2k,2k+1}, & \tau_2^{2k-1,2k} = \tau_2^{2k,2k+1}, & \sigma_3^{2k-1,2k} = \sigma_3^{2k,2k+1} \\ Q_1^{2k} = e^{2k} \tau_1^{2k,2k+1}, & Q_2^{2k} = e^{2k} \tau_2^{2k,2k+1} \end{cases} \quad (8)$$

where  $1 \leq k \leq N-1$ . It is worth mentioning that the 3 conditions in the first line in equation (8) are the same conditions required to model the thin layers as interfaces (continuity of the stress vector across an interface). The 5 conditions in equation (8) imply that the out-of-plane stresses in the thin layers are  $z$ -independent and this agrees with the assumption of Aquino et al [28] regarding the negligible variations of the out-of-plane stresses through the thickness of the interface.

The term  $\int_{h_{2k}^-}^{h_{2k}^+} \sigma_{op}^* \varepsilon_{op}^p dz$  in equation (4) and the stress approximation in the layers  $2k$  yield

the following generalized plastic strains:

$$\varepsilon_{\alpha\beta}^{2k,p} = \int_{h_{2k}^-}^{h_{2k}^+} \frac{\varepsilon_{\alpha\beta}^p}{e^{2k}} dz, \quad \chi_{\alpha\beta}^{2k,p} = \int_{h_{2k}^-}^{h_{2k}^+} \frac{12}{(e^{2k})^2} P_1^{2k}(z) \varepsilon_{\alpha\beta}^p dz, \quad d_{\Phi\alpha}^{2k,p} = 2 \int_{h_{2k}^-}^{h_{2k}^+} \frac{\varepsilon_{\alpha 3}^p}{e^{2k}} dz, \quad \varepsilon_{33}^{2k,p} = \int_{h_{2k}^-}^{h_{2k}^+} \frac{\varepsilon_{33}^p}{e^{2k}} dz \quad (9)$$

The variational property of the  $H.R.$  functional with respect to the generalized forces and moments yield the generalized constitutive equations related to:

- the in-plane force resultants in layer  $m$  ( $1 \leq m \leq 2N-1$ ):

$$\begin{aligned} \varepsilon_{\alpha\alpha}^m - \varepsilon_{\alpha\alpha}^{m,p} &= \frac{S_{\alpha\beta}^m N_{\beta\beta}^m + S_{\alpha 6}^m N_{12}^m}{e^i} + S_{C\alpha\alpha}^i \left( \frac{\sigma_3^{m,m+1} + \sigma_3^{m-1,m}}{4} \right) \\ 2(\varepsilon_{12}^m - \varepsilon_{12}^{m,p}) &= \frac{S_{\alpha 6}^m N_{\alpha\alpha}^m + S_{66}^m N_{12}^m}{e^i} + S_{C12}^m \left( \frac{\sigma_3^{m,m+1} + \sigma_3^{m-1,m}}{4} \right) \end{aligned} \quad (10)$$

- the in-plane moment resultants in layer  $m$  ( $1 \leq m \leq 2N-1$ ):

$$\begin{aligned} \chi_{\alpha\alpha}^m - \chi_{\alpha\alpha}^{m,p} &= \frac{12}{e^{m^3}} (S_{\alpha\beta}^m M_{\beta\beta}^m + S_{\alpha 6}^m M_{12}^m) + \frac{3}{5} S_{C\alpha\alpha}^m (\sigma_3^{m,m+1} - \sigma_3^{m-1,m}) \\ 2(\chi_{12}^m - \chi_{12}^{m,p}) &= \frac{12}{e^{m^3}} (S_{\alpha 6}^m M_{\alpha\alpha}^m + S_{66}^m M_{12}^m) + \frac{3}{5} S_{C12}^m (\sigma_3^{m,m+1} - \sigma_3^{m-1,m}) \end{aligned} \quad (11)$$

- the out-of-plane shear resultants in layer  $m$  ( $1 \leq m \leq 2N-1$ ):

$$d_{\Phi\alpha}^m - d_{\Phi\alpha}^{m,p} = \frac{6}{5e^m} S_{Q\alpha\beta}^m Q_\beta^m - \frac{1}{10} S_{Q\alpha\beta}^m (\tau_{\beta 3}^{m,m+1} + \tau_{\beta 3}^{m-1,m}) \quad (12)$$

where  $\varepsilon_{\alpha\beta}^{m,p} = \chi_{\alpha\beta}^{m,p} = d_{\Phi\alpha}^{m,p} = 0$  if  $m$  is an odd number (it corresponds to a thick layer).

The variational property of the  $H.R.$  functional with respect to:

- $\tau_{\alpha 3}^{n,n+1}$  yields the following constitutive equation for  $1 \leq n \leq 2N-2$ :

$$D_\alpha^{n,n+1} = -\frac{1}{10} S_{Q\alpha\beta}^n Q_\beta^n - \frac{1}{10} S_{Q\alpha\beta}^{n+1} Q_\beta^{n+1} - \frac{e^n}{30} S_{Q\alpha\beta}^n \tau_{\beta 3}^{n-1,n} + \frac{2}{15} (e^n S_{Q\alpha\beta}^n + e^{n+1} S_{Q\alpha\beta}^{n+1}) \tau_{\beta 3}^{n,n+1} - \frac{e^{n+1}}{30} S_{Q\alpha\beta}^{n+1} \tau_{\beta 3}^{n+1,n+2} \quad (13)$$

- $\sigma_3^{n,n+1}$  yields the following constitutive equation for  $1 \leq n \leq 2N-2$ :

$$D_3^{n,n+1} - D_3^{n,n+1,p} = \frac{9}{70} e^n S_3^n \sigma_3^{n-1,n} + \frac{13}{35} (e^n S_3^n + e^{n+1} S_3^{n+1}) \sigma_3^{n,n+1} + \frac{9}{70} e^{n+1} S_3^{n+1} \sigma_3^{n+1,n+2} + \frac{1}{4} (S_{c\alpha\beta}^n N_{\alpha\beta}^n + S_{c\alpha\beta}^{n+1} N_{\alpha\beta}^{n+1}) + \frac{3}{5} \left( \frac{S_{c\alpha\beta}^n M_{\alpha\beta}^n}{e^n} - \frac{S_{c\alpha\beta}^{n+1} M_{\alpha\beta}^{n+1}}{e^{n+1}} \right) \quad (14)$$

where  $D_3^{2k,2k+1,p} = \frac{e^{2k}}{2} \varepsilon_{33}^{2k,p}$  and  $D_3^{2k-1,2k,p} = \frac{e^{2k}}{2} \varepsilon_{33}^{2k,p}$  for  $1 \leq k \leq N-1$ .

Let us point out that in equations (10), (11) and (14), the coupling between the peel stresses and the in-plane forces is taken into account whereas the original M4-5N model did not (the compliance  $S_{c\alpha\beta}^i$  did not appear in the original equations). Most models of plates do not take into account this coupling and implies neglecting the Poisson's effect through the thickness direction. This is a first improvement of the M4-5N model.

Now, let us determine the generalized equations that will help to obtain the generalized plastic strains in the thin layer  $2k$  appearing in equation (9). For sake of simplicity, let us consider the case of associative plasticity with a normal flow rule, an isotropic hardening and a Von-Mises-like yield function. The 3D plastic strains are obtained by making use of

- the yield function

$$f(\boldsymbol{\sigma}, p) = \sigma^{eq} - R^{2k}(p) - R_0^{2k} \quad (15)$$

where  $\sigma^{eq}$  is the equivalent Von-Mises stress,  $R^{2k}$  is the hardening function,  $p$  is the cumulative plastic strain,  $R_0^{2k}$  is the initial yield stress.

- the flow rule

$$\dot{\varepsilon}_{op}^p = \frac{3}{2} \dot{p} \frac{\sigma_{op}^d}{R^{2k} + R_0^{2k}} \quad (16)$$

where the upper dot denotes the increment of a field with respect to its value in the previous load step,  $\sigma^d$  is the deviatoric stress tensor.

Let us now introduce the stress approximation of the M4-5N model in equations (15) and (16). In the thin layer  $2k$ , we assume that the contribution of the moments  $M_{\alpha\beta}^{2k}$  on the equivalent stress is negligible as compared to those of the other generalized forces and stresses. This equivalent stress is then uniform through the thickness of the thin layer. By making use of the yield function in equation (15), we prove that the cumulative plastic strain is also uniform through the thickness of the thin layer.

In the layerwise modelling, the cumulative plastic strain in each layer  $2k$  is then approximated by the following polynomial:

$$p(x, y, z) = p^{2k}(x, y) P_0^{2k}(z) = p^{2k}(x, y) \quad (17)$$

where  $p^{2k}$  is the generalized cumulative plastic strain in layer  $2k$ . The generalized yield function of layer  $2k$  is then:

$$f^{2k}(\sigma^{2k}, p^{2k}) = \sigma^{2k} - R^{2k}(p^{2k}) - R_0^{2k} \quad (18)$$

where  $\sigma^{2k}$  is the equivalent stress using the following stress approximation in layer  $2k$ :

$$\sigma_{\alpha\beta}^{2k} = \frac{N_{\alpha\beta}^{2k}}{e^{2k}}, \quad \sigma_{\alpha 3}^{2k} = \tau_{\alpha 3}^{2k, 2k+1}, \quad \sigma_{33}^{2k} = \sigma_3^{2k, 2k+1} \quad (19)$$

The generalized flow rules which provide the increments of the generalized plastic strains defined in (9) are then:

$$\begin{aligned}
\dot{\varepsilon}_{11}^{2k} &= \dot{\Delta}^{2k} \left[ \frac{2N_{11}^{2k} - N_{22}^{2k}}{3e^{2k}} - \frac{\nu^{2k,2k+1}}{3} \right], & \dot{\varepsilon}_{22}^{2k} &= \dot{\Delta}^{2k} \left[ \frac{2N_{22}^{2k} - N_{11}^{2k}}{3e^{2k}} - \frac{\nu^{2k,2k+1}}{3} \right], \\
\dot{\varepsilon}_{12}^{2k} &= \dot{\Delta}^{2k} \frac{N_{12}^{2k}}{e^{2k}}, & \dot{\chi}_{\alpha\beta}^{2k} &= 0, & \dot{\varepsilon}_{\alpha 3}^{2k} &= \dot{\Delta}^{2k} \frac{Q_{\alpha}^{2k}}{e^{2k}}, & \dot{\varepsilon}_{33}^{2k} &= \dot{\Delta}^{2k} \left[ \frac{2\nu^{2k,2k+1}}{3} - \frac{N_{22}^{2k} + N_{11}^{2k}}{3e^{2k}} \right]
\end{aligned} \quad (20)$$

$$\text{where } \dot{\Delta}^{2k} = \frac{3}{2} \frac{\dot{p}^{2k}}{R^{2k}(p^{2k}) + R_0^{2k}}$$

Let us point out that any other yield function may be applied. In what follows, for simplicity sake, the previously defined isotropic hardening and Von-Mises-like yield function are considered.

## 2. Enhanced model of laminates with imperfect interfaces

Let us now apply the equations in the previous section to determine the equations of a laminate with imperfect interfaces. If the aim is to model the thin layers as imperfect interfaces, a new renumbering of layers is required. Thick layers are numbered from 1 to  $N$ . The equations developed in the previous section are applied but the fields are to be renumbered. The renumbering is performed with the following method:

- for the generalized fields of the thick layers of the previous section the upper index  $2l-1$  is replaced by  $i$ ; for example:  $N_{11}^{2l-1}$  becomes  $N_{11}^i$ ,
- for the generalized stresses at the interfaces,  $\tau_{\alpha 3}^{2k-1,2k} = \tau_{\alpha 3}^{2k,2k+1}$  and  $\sigma_3^{2k-1,2k} = \sigma_3^{2k,2k+1}$  are replaced by  $\tau_{\alpha 3}^{j,j+1}$  and  $\sigma_3^{j,j+1}$ ,
- for the generalized fields of the thin layers, the upper index  $2k$  is replaced by “ $j,j+1$ ”; for instance:  $N_{11}^{2k}$  becomes  $N_{11}^{j,j+1}$ ,
- thicknesses  $e^{2l-1}$  and  $e^{2k}$  are replaced by  $t^i$  and  $\theta^{j,j+1}$ , respectively,

where  $1 \leq l \leq N$ ,  $1 \leq i \leq N$ ,  $1 \leq k \leq N-1$  and  $1 \leq j \leq N-1$ . Let us point out that  $\sigma_3^{j,j+1}$  and  $\tau_{\alpha 3}^{j,j+1}$  defined above are the generalized interfacial stresses at the imperfect interfaces.

In the “interface layer”, the contribution of the generalized moments on the stress field is neglected as compared to that of the other generalized forces and stresses. The material in this “interface layer” between layers  $j$  and  $j+1$  is isotropic and its properties are  $E^{j,j+1}$  (Young’s modulus),  $\nu^{j,j+1}$  (Poisson’s ratio),  $R^{j,j+1}$  (hardening function) and  $R_0^{j,j+1}$  (initial yield stress).

In the  $N$  layers, called thick layers in the previous section, a stress approximation equivalent to that in the previous section is adopted. Generalized forces ( $N_{\alpha\beta}^i$  and  $Q_\alpha^i$ ), stresses ( $\tau_{\alpha\beta}^{j,j+1}$  and  $\sigma_3^{j,j+1}$ ) and moments ( $M_{\alpha\beta}^i$ ) similar to those shown in equations (2) are defined.

The generalized displacements  $U_o^i$  and  $\Phi_\alpha^i$  similar to those in equations (5) are defined. Also the following generalized strains are defined:

$$\begin{aligned}\varepsilon_{\alpha\beta}^i(x, y) &= \frac{1}{2}(U_{\alpha,\beta}^i + U_{\beta,\alpha}^i), \quad \chi_{\alpha\beta}^i(x, y) = \frac{1}{2}(\Phi_{\alpha,\beta}^i + \Phi_{\beta,\alpha}^i), \\ d_{\Phi_\alpha}^i(x, y) &= \Phi_\alpha^i + U_{3,\alpha}^i, \quad D_\alpha^{j,j+1}(x, y) = U_\alpha^{j+1} - U_\alpha^j - \frac{t^j}{2}\Phi_\alpha^j - \frac{t^{j+1}}{2}\Phi_\alpha^{j+1} \\ \text{and } D_3^{j,j+1}(x, y) &= U_3^{j+1} - U_3^j\end{aligned}\quad (21)$$

Let us also define the generalized in-plane stresses  $\sigma_\alpha^{j,j+1}$ ,  $\sigma_{12}^{j,j+1}$  and displacements  $U_o^{j,j+1}$  at the “interface layer” between layers  $j$  and  $j+1$  by:

$$\begin{aligned}\sigma_\alpha^{j,j+1} &= \frac{1}{\theta^{j,j+1}} \int_{\text{interface thickness}} \sigma_{\alpha\alpha} dz, \quad \sigma_{12}^{j,j+1} = \frac{1}{\theta^{j,j+1}} \int_{\text{interface thickness}} \sigma_{12} dz \\ \text{and } U_o^{j,j+1} &= \frac{1}{\theta^{j,j+1}} \int_{\text{interface thickness}} U_o dz\end{aligned}\quad (22)$$

These last displacements do not appear in the definitions of the generalized strains in equation (21). The generalized equilibrium equations are:

$$\begin{cases} N_{\alpha\beta,\beta}^i + \tau_\alpha^{i,i+1} - \tau_\alpha^{i-1,i} = 0 \\ Q_{\alpha,\alpha}^i + \sigma_3^{i,i+1} - \sigma_3^{i-1,i} = 0 \\ M_{\alpha\beta,\beta}^i - Q_\alpha^i + \frac{t^i}{2}(\tau_\alpha^{i,i+1} + \tau_\alpha^{i-1,i}) = 0 \end{cases}\quad (23)$$

The generalized plastic strains at interface  $j, j+1$  are deduced from equation (9):

$$\varepsilon_{op}^{j,j+1 p} = \int_{\text{interface thickness}} \frac{\varepsilon_{op}^p}{\theta^{j,j+1}} dz \quad (24)$$

Let us point out that the generalized plastic strain  $\chi_{\alpha\beta}^{2k p}$  in equation (9) is not taken into account anymore because the generalized flow rule in the previous section yields  $\dot{\chi}_{\alpha\beta}^{2k p} = 0$ .

The generalized constitutive equations in the layers are deduced from equations (10-12) by replacing  $m$  by  $i$  ( $1 \leq i \leq N$ ) and substituting  $E^{j,j+1}$  and  $\nu^{j,j+1}$  in the compliances of the isotropic adhesives. The generalized constitutive equations at the interface between layers  $j$  and  $j+1$ :

- related to the in-plane stresses are

$$\begin{aligned} U_{1,1}^{j,j+1} - \varepsilon_{11}^{j,j+1 p} &= \frac{\sigma_1^{j,j+1}}{E^{j,j+1}} - \frac{\nu^{j,j+1}}{E^{j,j+1}} (\sigma_2^{j,j+1} + \sigma_3^{j,j+1}) \\ U_{2,2}^{j,j+1} - \varepsilon_{22}^{j,j+1 p} &= \frac{\sigma_2^{j,j+1}}{E^{j,j+1}} - \frac{\nu^{j,j+1}}{E^{j,j+1}} (\sigma_1^{j,j+1} + \sigma_3^{j,j+1}) \\ U_{1,2}^{j,j+1} + U_{2,1}^{j,j+1} - 2\varepsilon_{12}^{j,j+1 p} &= \frac{2(1 + \nu^{j,j+1})}{E^{j,j+1}} \sigma_{12}^{j,j+1} \end{aligned} \quad (25)$$

- related to the interfacial shear stresses are

$$\begin{aligned} D_\alpha^{j,j+1} - \gamma_\alpha^{j,j+1} &= -\frac{1}{10} S_{Q\alpha\beta}^j Q_\beta^j - \frac{1}{10} S_{Q\alpha\beta}^{j+1} Q_\beta^{j+1} - \frac{t^j}{30} S_{Q\alpha\beta}^j \tau_{\beta 3}^{j-1,j} \\ &\quad + \frac{2}{15} (t^j S_{Q\alpha\beta}^j + t^{j+1} S_{Q\alpha\beta}^{j+1}) \tau_{\beta 3}^{j,j+1} - \frac{t^{j+1}}{30} S_{Q\alpha\beta}^{j+1} \tau_{\beta 3}^{j+1,j+2} \end{aligned} \quad (26)$$

where  $\gamma_\alpha^{j,j+1}$  is the generalized in-plane displacement discontinuity field defined by

$$\begin{aligned} \gamma_\alpha^{j,j+1} &= \frac{2\theta^{j,j+1} \tau_{13}^{j,j+1}(x,y)(1 + \nu^{j,j+1})}{E^{j,j+1}} - g_\alpha^{j,j+1} + \gamma_\alpha^{j,j+1 p}, \\ g_\alpha^{j,j+1} &= \theta^{j,j+1} U_{3,\alpha}^{j,j+1} \quad \text{and} \quad \gamma_\alpha^{j,j+1 p} = 2\theta^{j,j+1} \varepsilon_{\alpha 3}^{j,j+1 p} \end{aligned} \quad (27)$$

- related to the interfacial peel stresses are

$$\begin{aligned} D_3^{j,j+1} - \gamma_3^{j,j+1} &= \frac{9}{70} t^j S_3^j \sigma_3^{j-1,j} + \frac{13}{35} (t^j S_3^j + t^{j+1} S_3^{j+1}) \sigma_3^{j,j+1} + \frac{9}{70} t^{j+1} S_3^{j+1} \sigma_3^{j+1,j+2} \\ &\quad + \frac{1}{4} (S_{c\alpha\beta}^j N_{\alpha\beta}^j + S_{c\alpha\beta}^{j+1} N_{\alpha\beta}^{j+1}) + \frac{3}{5} \left( \frac{S_{c\alpha\beta}^j M_{\alpha\beta}^j}{t^j} - \frac{S_{c\alpha\beta}^{j+1} M_{\alpha\beta}^{j+1}}{t^{j+1}} \right) \end{aligned} \quad (28)$$

where  $\gamma_3^{j,j+1}$  is the generalized out-of-plane displacement discontinuity field defined by

$$\gamma_3^{j,j+1} = \theta^{j,j+1} \left( \frac{\sigma_3^{j,j+1}}{E^{j,j+1}} - \frac{\nu^{j,j+1}}{E^{j,j+1}} \sigma_\alpha^{j,j+1} \right) + \gamma_3^{j,j+1p} \text{ and } \gamma_3^{j,j+1p} = \theta^{j,j+1} \varepsilon_{33}^{j,j+1p}. \quad (29)$$

Equation (26) is obtained by making use of equations (12-13) and summing  $D_\alpha^{2k-1,2k} - D_\alpha^{2k-1,2k p}$  and  $D_\alpha^{2k,2k+1} - D_\alpha^{2k,2k+1 p}$ . Equation (28) is obtained by making use of equation (14) and summing  $D_3^{2k-1,2k} - D_3^{2k-1,2k p}$  and  $D_3^{2k,2k+1} - D_3^{2k,2k+1 p}$ .

The expressions of the generalized displacements  $U_o^{j,j+1}$  of the “interface layers” are obtained by applying equations (13-14) and calculating the sum of  $D_o^{2k-1,2k} - D_o^{2k-1,2k p}$  and  $D_o^{2k,2k+1 p} - D_o^{2k,2k+1}$  ( $1 \leq o \leq 3$ ) in the numbering rule of the previous section. With the new numbering rule, calculations yield:

$$\begin{aligned} 2U_3^{j,j+1} = & U_3^j + U_3^{j+1} + \frac{9}{70} t^j S_3^j \sigma_3^{j-1,j} + \frac{13}{35} (t^j S_3^j - t^{j+1} S_3^{j+1}) \sigma_3^{j,j+1} - \frac{9}{70} t^{j+1} S_3^{j+1} \sigma_3^{j+1,j+2} \\ & + \frac{1}{4} (S_{c\alpha\beta}^j N_{\alpha\beta}^j - S_{c\alpha\beta}^{j+1} N_{\alpha\beta}^{j+1}) + \frac{3}{5} \left( \frac{S_{c\alpha\beta}^j M_{\alpha\beta}^j}{t^j} + \frac{S_{c\alpha\beta}^{j+1} M_{\alpha\beta}^{j+1}}{t^{j+1}} \right) \end{aligned} \quad (30)$$

and

$$\begin{aligned} 2U_\alpha^{j,j+1} = & U_\alpha^{j+1} + U_\alpha^j + \frac{t^j}{2} \Phi_\alpha^j - \frac{t^{j+1}}{2} \Phi_\alpha^{j+1} + \frac{1}{10} S_{Q\alpha\beta}^{j+1} Q_\beta^{j+1} - \frac{1}{10} S_{Q\alpha\beta}^j Q_\beta^j - \frac{t^j}{30} S_{Q\alpha\beta}^j \tau_{\beta 3}^{j-1,j} \\ & + \frac{2}{15} (t^j S_{Q\alpha\beta}^j - t^{j+1} S_{Q\alpha\beta}^{j+1}) \tau_{\beta 3}^{j,j+1} + \frac{t^{j+1}}{30} S_{Q\alpha\beta}^{j+1} \tau_{\beta 3}^{j+1,j+2} \end{aligned} \quad (31)$$

The generalized plastic displacement discontinuities  $\gamma_o^{j,j+1p}$  at interface between layers  $j$  and  $j+1$  appearing in equations (27) and (29) are obtained by making use of the following equations:

- the generalized yield function of interface  $j, j+1$

$$f^{j,j+1}(\sigma^{j,j+1}, p^{j,j+1}) = \sigma^{j,j+1} - R^{j,j+1}(p^{j,j+1}) - R_0^{j,j+1} \quad (32)$$



where  $p^{j,j+1}$  is the generalized cumulative plastic strain of the interface,  $\sigma^{j,j+1}$  is the equivalent Von-Mises stress obtained for the following stress field components

$$\sigma_1^{j,j+1}, \sigma_2^{j,j+1}, \sigma_3^{j,j+1}, \sigma_{12}^{j,j+1}, \tau_{13}^{j,j+1}, \tau_{23}^{j,j+1} \quad (33)$$

- the generalized flow rules of interface  $j, j+1$

$$\text{if } \begin{cases} f^{j,j+1} = 0 \\ \dot{f}^{j,j+1} \geq 0 \end{cases} \text{ then } \begin{cases} \dot{\varepsilon}_{11}^{j,j+1} = \dot{\Delta}^{j,j+1} \left[ \frac{2\sigma_1^{j,j+1} - \sigma_2^{j,j+1} - \sigma_3^{j,j+1}}{3} \right], \\ \dot{\varepsilon}_{22}^{j,j+1} = \dot{\Delta}^{j,j+1} \left[ \frac{2\sigma_2^{j,j+1} - \sigma_1^{j,j+1} - \sigma_3^{j,j+1}}{3} \right], \\ \dot{\varepsilon}_{33}^{j,j+1} = \dot{\Delta}^{j,j+1} \left[ \frac{2\sigma_3^{j,j+1} - \sigma_2^{j,j+1} - \sigma_1^{j,j+1}}{3} \right], \\ \dot{\varepsilon}_{12}^{j,j+1} = \dot{\Delta}^{j,j+1} \sigma_{12}^{j,j+1}, \quad \dot{\varepsilon}_{\alpha 3}^{j,j+1} = \dot{\Delta}^{j,j+1} \tau_{\alpha 3}^{j,j+1} \end{cases} \quad (34)$$

$$\text{where } \dot{\Delta}^{j,j+1} = \frac{3}{2} \frac{\dot{p}^{j,j+1}}{R^{j,j+1}(p^{j,j+1}) + R_0^{j,j+1}}.$$

Let us describe in this paragraph the main differences of the present model with that developed by Aquino de los Rios et al [28]. In [28],

- the Poisson's effect through the thickness direction is neglected. Similar equations to those in equations (10-11) and (28) are obtained but the compliances  $S_{C\alpha\beta}^i$  are replaced by zero
- the in-plane stresses  $\sigma_1^{j,j+1}$ ,  $\sigma_2^{j,j+1}$  and  $\sigma_{12}^{j,j+1}$  at the “interface layer” are not considered. Equations (25) do not exist.
- the same elastic interfacial constitutive equations (26) and (28) are obtained but the displacements  $U_1^{j,j+1}$ ,  $U_2^{j,j+1}$  and  $U_3^{j,j+1}$  at the interface are calculated by empirical equations involving an average of the displacements of the adjacent layers; for example:

$$U_3^{j,j+1} = \left( \frac{U_3^j(x, y)t^{j+1} + U_3^{j+1}(x, y)t^j}{t^j + t^{j+1}} \right)$$

This equation was proposed by making a sort of approximation of the 3D displacement which is not suitable since the starting point of the modelling is a stress approximation. The above equation is not as accurate as equation (30) because it does not take into account the stiffnesses of the adjacent layers.

- the yield function and the flow rule only take into account the interfacial shear and peel stresses.

Finally, let us point out that the resolution of the equations of the present model is performed by means of a similar numerical technique to that applied by Aquino de los Rios et al in [28]: a finite element resolution combined with a Newton-Raphson like method. Further details of the numerical technique are shown in [28].

### 3. Theoretical validation of the model

Let us now make a theoretical validation of the model by comparing its results to those of a solid finite element (FE) resolution performed by the commercial software called COMSOL Multiphysics 3.1. A first validation (not shown in this paper) consists on considering the case of a double-lap joint subjected to a tensile load (the same case was considered by Aquino de los Rios et al [28]). In this case, the adhesive is subjected essentially to a shear loading and the enhanced version of the model yields practically the same results as those of the previous one and both are very similar to the solid finite element results. For simplicity sake, this case is not shown in this paper. Let us now consider the case of a peel dominated loading. The structure considered is a T-peel joint (see Figure 2) with elastic steel substrates and an elastic-plastic adhesive which plastic behaviour is modelled by a Von-Mises yield function, a normal flow rule and an associative plasticity consideration. The width of the joint is 25mm. The material properties are shown in table 1. A plane strain state is assumed. In Figure 3, the mesh considered in the solid FE calculation is shown; let us point out the high number of elements required due to the important difference in thicknesses for the adhesive and the adherend. In order to apply our layerwise model to analysis of this joint, symmetry is applied as shown in Figure 2. At the left end of the adherend, a force  $F$  and a bending moment  $M = b \times F$  are applied. A monotonic load  $F$  is considered. According to

Castagnetti et al [30], this joint fails for a 487.5N load. A reasonable set of load values to be considered is then  $F=118\text{N}$  (when plasticity initiates for the layerwise model),  $F=300\text{N}$  and  $F=500\text{N}$ . The generalized stresses calculated by the M4-5N model at the “interface layer” are compared to the stresses calculated by COMSOL in the adhesive at the symmetry axis (the y-axis, see Figure 2). It is not worthy to use in this comparison the stresses calculated by COMSOL at the adhesive/steel interface because the left edge exhibits singularities.

In Figure 4, the equivalent Von-Mises stresses  $\sigma^{eq}$  in the adhesive calculated by the M4-5N model and COMSOL are plotted against the y coordinate for the three load levels considered. At a 118N, the equivalent stress calculated by the M4-5N model at the left edge reaches the yield stress and plasticity onset is predicted. For the three considered loads, an excellent agreement between the two calculation techniques is observed.

In Figure 5, the normal stresses  $\sigma_3$  in the adhesive calculated by the M4-5N model and COMSOL are plotted against the y coordinate. The stress calculated by M4-5N is in fact the generalized peel stress  $\sigma_3^{1,2}$  at the “interface layer”. Once again, a very good agreement between the two calculation techniques is observed. Let us point out that these stresses reach much higher values (up to 188MPa) than the yield stress (30MPa) because the presence of the other normal stresses  $\sigma_1^{1,2}$  and  $\sigma_2^{1,2}$  reduces the equivalent stress level (the hydrostatic stress does not affect the equivalent stress). In Figure 6, the normal stresses  $\sigma_2$  in the adhesive computed by the two calculation methods are plotted against the y coordinate. The solid FE (COMSOL) values at the left edge for this stress are zero ( $\sigma_2 = 0$ ) because of the free boundary condition. This condition is not verified by the  $\sigma_2^{1,2}$  stress of the M4-5N model since the adhesive is modelled by an interface. This causes a slight difference between the values of the normal stresses near the left edge. In spite of this, the M4-5N results are very accurate over a wide range in the adhesive. For the other stress  $\sigma_1$  in the adhesive, the layerwise model and COMSOL provide also very similar results.

Let us now compare the predictions of the opening displacement  $\gamma_3^{1,2}$  at the imperfect interface of the M4-5N model to the elongation  $\gamma_3$  in the thickness direction of the adhesive computed by COMSOL ( $\gamma_3 = -2\nu$  where  $\nu$  is the displacement of the interface in the  $z$ -direction according to the drawing in Figure 2). In Figure 7, these displacements are plotted against the  $y$  coordinate for the three load levels considered. Once again, very accurate results are observed for the M4-5N model.

Finally, let us compare the results of the new version of the M4-5N with those of the previous one developed by Aquino de los Rios et al in [28]. In the previous version of the M4-5N, the sole peel stress  $\sigma_3^{1,2}$  controls plasticity onset for the T-peel joint and  $\sigma_3^{1,2}$  cannot overpass the yield stress (the in-plane stresses  $\sigma_1^{1,2}$  and  $\sigma_2^{1,2}$  were not taken into account). In Figure 8, the normal stresses  $\sigma_3$  in the adhesive calculated by the two versions of the M4-5N model and COMSOL are plotted against the  $y$  coordinate for the 500N load. An important difference between the predicted peel stress values by the two versions is observed. A comparison of the predictions of the two models with those of COMSOL allows us to state that in the case of a peel dominated loading, the previous M4-5N model yields erratic results whereas the enhanced layerwise model provides very accurate results.

Additionally to the accuracy of the enhanced version of the model, the layerwise technique has the quality to perform the calculations faster (at least 30 times faster than the solid FE technique for the structure above considered) and with less memory requirements.

## Conclusion

To conclude, a layerwise model previously developed for the analysis of laminates with thin adhesive layers has been enhanced in order to predict accurately the interfacial stresses and strains. The elastoplastic adhesive layers are modeled by imperfect interfaces in order to obtain a cost effective tool. A theoretical rigorous support has been provided for the constitutive equations of elastoplastic interfaces. These constitutive equations were obtained by making use of an adaptation of the Hellinger-Reissner

functional and an asymptotic expansion method based on the small thickness of the adhesive layers. The equations of the model were solved by a numerical tool developed in a previous paper. The model was then applied to the calculation of stresses and displacement discontinuities across the imperfect interfaces in a T-peel joint. The model results were compared to those of a solid FE resolution. The enhanced layerwise model provided very accurate results, practically the same as those of the solid FE technique whereas the previous layerwise model with imperfect interfaces yielded erratic predictions.

The enhanced layerwise model with imperfect interfaces proves to be a suitable tool for the analysis of laminates or adhesive joints with thin layers of an elastic-plastic adhesive subjected to different loading conditions. An important saving in computing cost is achieved with the use of this model instead of solid finite elements. In part 2 of this work, the model predictions are compared to experimental measurements in elastoplastic adhesive joints. Moreover, pertinent failure criteria are developed to obtain accurate predictions of failure onset for different adhesive joints.

## **Acknowledgements**

The authors wish to acknowledge CONACYT (grant no. CB-2007-01 79467 002) and US AIR FORCE (award no. FA9550-09-1-0394) for their financial support.

## **References**

- [1] Carrera E, Ciuffreda A. A unified formulation to assess theories of multilayered plates for various bending problems. *Composite Structures* 2005;69(3):271-293.
- [2] Carrera E, Cinefra M, Nali P. MITC technique extended to variable kinematic multilayered plate elements. *Composite Structures* 2010;92(8):1888-1895.
- [3] Catapano A, Giunta G, Belouettar Z, Carrera E. Static analysis of laminated beams via a unified formulation. *Composite Structures* 2011;94(1):75-83.
- [4] Ferreira AJM, Carrera E, Cinefra M, Roque CMC, Polit O. Analysis of laminated shells by a sinusoidal shear deformation theory and radial basis functions collocation, accounting for through-the-thickness deformations. *Composites Part B: Engineering* 2011;42(5):1276-1284.

- [5] Reissner E. On a variational theorem in elasticity. *Journal of Mathematics and Physics* 1950;29:90-95.
- [6] Reissner E. On a certain mixed variational theory and a proposed application. *International Journal of Numerical Methods in Engineering* 1984;20(7):1366–1368.
- [7] Limam O, Foret G, Ehrlacher A. RC beams strengthened with composite material: a limit analysis approach and experimental study. *Composite Structures* 2003;59(4):467-472.
- [8] Limam O, Foret G, Ehrlacher A. RC two-way slabs strengthened with CFRP strips: experimental study and a limit analysis approach. *Composite Structures* 2003;60(4):467-471.
- [9] Dallot J, Sab K. Limit analysis of multi-layered plates. Part II: Shear effects. *Journal of the Mechanics and Physics of Solids* 2008;56(2):581-612.
- [10] Lebée A, Sab K. A Cosserat multiparticle model for periodically layered materials. *Mechanics Research Communications* 2010;37(3):293-297.
- [11] Carrera E. Theories and finite elements for multilayered, anisotropic, composite plates and shells. *Archives of Computational Methods in Engineering* 2002;9(2):87–140.
- [12] Reissner E, Stavsky Y. Bending and stretching of certain types of heterogeneous anisotropic elastic plate. *Journal of Applied Mechanics* 1961;28(3):402-408.
- [13] Srinivas S. A redefined analysis of composite laminates. *Journal of Sound and Vibration* 1973;30(4):495-507.
- [14] Carrera E. Mixed layer-wise models for multilayered plates analysis. *Composite Structures* 1998;43(1):57-70.
- [15] Carrera E. Layer-wise mixed models for accurate vibrations analysis of multilayered plates. *Journal of Applied Mechanics* 1998;65(4):820-828.
- [16] Demasi L. Mixed plate theories based on the generalized unified formulation. Part ii: Layerwise theories. *Composite Structures* 2009;87(1):12-22.
- [17] Pagano NJ. Stress fields in composite laminates. *International Journal of Solids and Structures* 1978;14(5):385-400.
- [18] Chabot A. Analyse des efforts à l'interface entre les couches des matériaux composites à l'aide des M4. Doctoral dissertation, ENPC, France, 1997.

- [19] Carreira RP, Caron JF, Diaz Diaz A. Model of multilayered materials for interface stresses estimation and validation by finite element calculations. *Mechanics of Materials* 2002;34(4):217-230.
- [20] Diaz Diaz A, Caron JF, Carreira RP. Software application for evaluating interfacial stresses in inelastic symmetrical laminates with free edges. *Composite Structures* 2002;58(2):195-208.
- [21] Reissner E. The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics* 1945;A:69-77.
- [22] Diaz Diaz A, Caron JF. Interface plasticity and delamination onset prediction. *Mechanics of Materials* 2006;38(7):648-663.
- [23] Lecomte-Grosbras P, Paluch B, Brieu M, De Saxcé G, Sabatier L. Interlaminar shear strain measurement on angle-ply laminate free edge using digital image correlation. *Composites Part A: Applied Science and Manufacturing*. 2009; 40(12):1911-1920.
- [24] Chataigner S, Caron JF, Duong VA, Diaz Diaz A. Experimental and numerical investigation of shear strain along an elasto-plastic bonded lap joint. *Construction and Building Materials* 2011;25(2):432-441.
- [25] Cognard JY, Davies P, Sohier L, Créac'hcadec R. A study of the non-linear behaviour of adhesively-bonded composite assemblies. *Composite Structures* 2006;76(1-2):34-46.
- [26] Wang CH, Chalkley P. Plastic yielding of a film adhesive under multiaxial stresses. *International Journal of Adhesion and Adhesives* 2000;20(2):155-164.
- [27] Lin YC, Chen XM, Zhang J. Uniaxial ratchetting behavior of anisotropic conductive adhesive film under cyclic tension. *Polymer Testing* 2011;30(1):8-15.
- [28] Aquino de Los Rios GS, Castañeda Balderas R, Duong VA, Chataigner S, Caron JF, Ehrlicher A, Foret G, Diaz Diaz A, Laminated materials with plastic interfaces: modeling and calculation. *Modeling and Simulation in Materials Science and Engineering* 2009;17(2):1-21.
- [29] Duong VA, Diaz Diaz A, Chataigner S, Caron JF. A layerwise finite element for multilayers with imperfect interfaces. *Composite Structures* 2011;93(12):3262-3271.

[30] Castagnetti D, Dragoni E, Spaggiari A. Failure analysis of bonded T-peel joints: Efficient modelling by standard finite elements with experimental validation. *International Journal of Adhesion and Adhesives* 2010;30(5):306-312.



Figure 1. Laminated structure considered and two steps for its modelling.

Figure 2. T-peel joints: geometry a) and modeling b) (dimensions are in mm).

Figure 3. Meshed geometry in COMSOL.

Figure 4. Equivalent  $\sigma^{\text{eq}}$  stress in the adhesive for three loads (118, 300 and 500N).

Figure 5.  $\sigma_3$  stress in the adhesive for three loads (118, 300 and 500N).

Figure 6.  $\sigma_2$  stress in the adhesive for three loads (118, 300 and 500N).

Figure 7. Elongation  $\gamma_3$  in the thickness direction of the adhesive layer for three loads  
(118, 300 and 500N).

Figure 8.  $\sigma_3$  stress in the adhesive for a 500N load calculated by COMSOL and the two versions of the M4-5N model.

Property \ Material	Adhesive	Adherend
Young's modulus (GPa)	1.7	206
Poisson's ratio	0.3	0.3
Yield stress (MPa)	30	

Table 1. Properties of involved materials.